

Three Random Tangents to a Circle

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ABSTRACT. Among several things, we find the side density for random triangles circumscribing the unit circle and calculate that its median is 5.5482.... An analogous exact computation for perimeter density remains open.

Let us initially discuss two random tangents to the unit circle. Without loss of generality, let one of the tangents be the vertical line passing through the point $(-1, 0)$. Let the other tangent pass through the point $(\cos(\theta), \sin(\theta))$, where θ is uniformly distributed on the interval $[0, \pi]$. Hence it has slope $-\cot(\theta)$ and is the unique such line touching the upper semicircle. The two lines cross at (x, y) , where

$$\begin{cases} x = -1, \\ y - \sin(\theta) = -\cot(\theta)(x - \cos(\theta)) \end{cases}$$

thus $h = \cot(\theta/2)$ is the (positive) height of the intersection point. We wish to determine the probability density of h . Since

$$\frac{d}{d\theta} \cot\left(\frac{\theta}{2}\right) = -\frac{1}{2} \csc\left(\frac{\theta}{2}\right)^2$$

the density is [1]

$$\left. \frac{\frac{1}{\pi}}{\frac{1}{2} \csc\left(\frac{\theta}{2}\right)^2} \right|_{\theta=2 \arccot(h)} = \left. \frac{\frac{2}{\pi}}{\cot\left(\frac{\theta}{2}\right)^2 + 1} \right|_{\theta=2 \arccot(h)} = \frac{2}{\pi} \frac{1}{h^2 + 1}$$

for $h > 0$, that is, the one-sided Cauchy distribution. The mean of h is infinite, as is well-known; its median is 1.

Let us now discuss three random tangents to the unit circle, incorrectly modeled. Take first and second lines exactly as in the preceding, and define similarly a third line to touch the lower semicircle, independent of the second. We study $h + k$, where h is as before and k is the (positive) depth of the intersection between first and third lines. Let $\ell = h + k$. The density of ℓ is the convolution [1]

$$\frac{4}{\pi^2} \int_0^\ell \frac{1}{(\ell - k)^2 + 1} \frac{1}{k^2 + 1} dk = \frac{8}{\pi^2} \frac{\ell \arctan(\ell) + \ln(\ell^2 + 1)}{(\ell^2 + 4)\ell}$$

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for $\ell > 0$. Why is this model incorrect? Clearly $E(\theta) = \pi/2$, not $\pi/3$, hence the three contact points are not equidistant (on average). Consider also the triangle T determined by the three lines: ℓ is the vertical side of T , but cannot be regarded as an “arbitrary” side. The assumption that $\theta \sim \text{Uniform}[0, \pi]$ requires change.

Another change (less a requirement than a preference) involves the relationship between T and the unit circle C . Clearly C is an incircle of T if and only if there is no semicircle containing all three contact points. Otherwise C is an excircle of T . We wish to refine our model (which presently incorporates both incircles and excircles) so that the density of ℓ is based on incircles alone. Naturally $\ell > 2$; the infimum 2 occurs in the limit as second and third lines both become horizontal. The density for this refined model is given in the next section; a related optimization problem appears at the end. As far as we know, these results have not appeared in the random triangle literature before [2].

1. UNIT INRADIUS

Without loss of generality, let the first tangent be the vertical line passing through the point $(-1, 0)$. Let the second tangent pass through the point $(\cos(\alpha), \sin(\alpha))$; let the third tangent pass through the point $(\cos(\beta), -\sin(\beta))$. It is assumed that the bivariate density for angles α, β is

$$\begin{cases} 2/\pi^2 & \text{if } 0 < \alpha < \pi, 0 < \beta < \pi \text{ and } \alpha + \beta < \pi, \\ 0 & \text{otherwise.} \end{cases}$$

It is best to think of α being measured in a counterclockwise direction (as is customary) and β being measured in a clockwise direction. The condition $\alpha + \beta < \pi$ prevents contact points from all crowding onto any semicircle (think of what happens when $\alpha + \beta = \pi$). Dependency between α and β makes our analysis more complicated than earlier.

As a check, the univariate density for α is

$$\begin{cases} 2(\pi - \alpha)/\pi^2 & \text{if } 0 < \alpha < \pi, \\ 0 & \text{otherwise.} \end{cases}$$

Thus points on C far away from $(-1, 0)$ are favored (that is, small angles α are weighted more heavily than large α) and $E(\alpha) = \pi/3$.

We wish to determine the bivariate density of $h = \cot(\alpha/2)$, $k = \cot(\beta/2)$. Via a Jacobian determinant argument, the density is [1]

$$\frac{\frac{2}{\pi^2}}{\frac{1}{2} \csc\left(\frac{\alpha}{2}\right)^2 \frac{1}{2} \csc\left(\frac{\beta}{2}\right)^2} \bigg|_{\substack{\alpha=2 \operatorname{arccot}(h), \\ \beta=2 \operatorname{arccot}(k)}} = \frac{8}{\pi^2} \frac{1}{h^2 + 1} \frac{1}{k^2 + 1}$$

for $h > 0$, $k > 0$, $h k > 1$. The latter inequality is true because

$$\pi = \alpha + \beta = 2 \operatorname{arccot}(h) + 2 \operatorname{arccot}(k)$$

if and only

$$k = \cot \left[\frac{1}{2} (\pi - 2 \operatorname{arccot}(h)) \right] = \frac{1}{h}.$$

Let $\ell = h + k$. The density of ℓ is the convolution

$$\frac{8}{\pi^2} \int_{a(\ell)}^{b(\ell)} \frac{1}{(\ell - k)^2 + 1} \frac{1}{k^2 + 1} dk$$

where limits of integration $a(\ell)$, $b(\ell)$ are found from $(\ell - k)k > 1$, hence $k^2 - \ell k + 1 < 0$; the zeroes of the quadratic are

$$a(\ell) = \frac{\ell - \sqrt{\ell^2 - 4}}{2} > 0, \quad b(\ell) = \frac{\ell + \sqrt{\ell^2 - 4}}{2} < \ell$$

and these are real because $\ell > 2$. Integrating, we obtain the density of ℓ to be

$$\frac{16}{\pi^2} \frac{f(\ell) + g(\ell)}{(\ell^2 + 4)\ell}$$

for $\ell > 2$, where

$$f(\ell) = \ell \arctan \left(\frac{\ell + \sqrt{\ell^2 - 4}}{2} \right) - \ell \arctan \left(\frac{\ell - \sqrt{\ell^2 - 4}}{2} \right),$$

$$g(\ell) = \ln \left(\ell + \sqrt{\ell^2 - 4} \right) - \ln \left(\ell - \sqrt{\ell^2 - 4} \right).$$

The mean of ℓ is infinite; its median 5.5482039188784452776442997... can be computed to high numerical precision as a consequence of our exact density formula. See [3] for experimental confirmation of our work.

Having derived the density of an arbitrary side, let us briefly mention other properties. The triangle T , under the condition that it circumscribes the circle C , is acute with probability $1/4$ [4, 5]. Since the inradius of T is 1, the area of T is one-half the perimeter of T [6, 7]. Unfortunately the perimeter density of T (as well as a trivariate density for sides) remains analytically intractable.

2. OPTIMIZATION PROBLEM

Of all triangles circumscribing the unit circle, an equilateral triangle minimizes the perimeter p . The minimum value for p is $6\sqrt{3} = 10.39230\dots$

Let s denote a side of a triangle of unit inradius. If the other two sides are nearly parallel and infinite, then s approaches 2 from above. The infimum for s is 2.

Let u, v denote two sides of a triangle of unit inradius. What is the minimum value for $u + v$? It is surprising that this question is not better known, especially since the answers for one side (s) and for the sum of three sides (p) are clear.

By symmetry, the minimizing triangle is isosceles and $u = v$. Let w denote the remaining side of the triangle. By Heron's formula,

$$u + v + w = \frac{1}{2} \sqrt{(u + v + w)(-u + v + w)(u - v + w)(u + v - w)}$$

hence

$$4(u + v + w) = (-u + v + w)(u - v + w)(u + v - w)$$

hence

$$4(2v + w) = w^2(2v - w)$$

hence

$$v = \frac{(w^2 + 4)w}{2(w^2 - 4)}.$$

Differentiating with respect to w , we find that

$$w = \sqrt{8 + 4\sqrt{5}}$$

is a zero of the derivative. Substituting into the expression for v , we deduce that the minimum value for $u + v$ is

$$2v = \sqrt{22 + 10\sqrt{5}} = 6.66038\dots$$

The angle θ at the apex of the minimizing triangle is also interesting. By the Law of Cosines,

$$u^2 + v^2 - 2uv \cos(\theta) = w^2$$

hence

$$2v^2 - w^2 = 2v^2 \cos(\theta)$$

hence

$$\cos(\theta) = 1 - \frac{1}{2} \frac{w^2}{v^2} = -2 + \sqrt{5} = \frac{1}{\varphi^3}$$

where φ is the Golden mean [8]. Finally, $\theta = 1.33247\dots \approx 76.34^\circ$. This material constitutes a (very small) first step toward characterizing the density for the sum of two arbitrary sides of T .

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